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## The semiclassical limit of the spin boson model

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**Abstract.** We study the semiclassical limit of the spin boson model, where the spin remains quantized and the field is expanded around the classical limit of a large number of photons, using a phase representation due to Bialynicki-Birula and Bialynicki-Birula. We derive the semiclassical Hamiltonian and prove that in the semiclassical expansion there always arises a back reaction of the atom on the field. The nearest-neighbour spacing distribution of levels is studied in the same limit and shown to be, in general, qualitatively different from the distribution for the original model. However, the comparison between the Dyson–Mehta statistics is also studied and no qualitative difference is visible for the function  $\Delta_3$ .

### 1. Introduction and summary

The spin boson model is a very interesting model in physics, with a large variety of applications [1]. In this paper we shall be interested in the applications to quantum optics, where the spin represents a two-level atom (for spin- $\frac{1}{2}$ ) and the boson the electromagnetic field. In a recent paper [2] we studied the classical limit of the model. The classical equations of motion of [3], which exhibit chaotic behaviour for large coupling constant, are obtained from the Heisenberg equations of motion for the quantum model, upon requiring the usual ‘factorization property’ (see [1] and references therein). The latter property is rigorously derivable from the  $N \rightarrow \infty$  limit of an  $N$ -atom Hamiltonian, in a precise sense [4], reviewed in the first part of [2], in which both the atoms and the field become classical. This property is therefore harder to justify—in spite of its thorough use—for the micromaser [5], where a single two-level atom interacts with the field of a single mode lossless resonator. This led us to study what we call the semiclassical limit of the model, where the spin remains quantized and the field is expanded around the classical limit for the beam, which corresponds to a large number  $n_0$  of photons [6].

In order to evaluate most conveniently the effects of the counter-rotating terms, we consider the following Hamiltonian:

$$H = \omega a^\dagger a + \omega_0 S_z + \lambda[(S^+ a + S^- a^\dagger) + \varepsilon(S^+ a^\dagger + S^- a)] \quad (1.1)$$

on the tensor product  $\mathbb{C}^2 \otimes \mathcal{F}$ , where  $S_x, S_y, S_z$  are spin- $\frac{1}{2}$  operators satisfying  $[S_x, S_y] = iS_z$  (with cyclic permutations),  $S^\pm = (S_x \pm iS_y)/2$ ;  $a, a^\dagger$  are standard annihilation and creation operators with  $[a, a^\dagger] = \mathbb{1}$  acting on Fock space  $\mathcal{F}$ , and the frequencies  $\omega, \omega_0$  ( $\hbar = 1$ ) and the coupling  $\lambda$  are real constants, which we take to be positive. The

anisotropy parameter  $0 \leq \varepsilon \leq 1$  interpolates between the Jaynes-Cummings model [7] ( $\varepsilon = 0$ ), where the rotating-wave approximation is made, and which is therefore classically integrable, and the spin boson model ( $\varepsilon = 1$ ) which is classically chaotic for large  $\lambda$ . The parameter  $\varepsilon$  is therefore a suitable measure of the deviation from integrability.

This paper is organized as follows. In section 2 we study (1.1) (for simplicity only for  $\varepsilon = 1$ ) in the phase representation for intense photon beams (i.e. with a large number  $n_0$  of photons) due to Bialynicki-Birula and Bialynicki-Birula [8], which allows an expansion of the creation and annihilation operators in powers of  $n_0$ . Our main result is the derivation of the semiclassical Hamiltonian. It is thereby seen that the Autler-Townes Hamiltonian [9]—which yields a linear non-autonomous set of equations of motion without chaotic behaviour [1]—is not obtained from the semiclassical expansion of the quantum Hamiltonian: it does not correspond to the first-order term, while, in second order, further terms arise in the expansion which preserve the nonlinear character of the equations of motion and represent a back reaction of the atoms on the field. The physical situation without back reaction which is described by Autler and Townes is, nevertheless, realizable, and it may be asked whether it may be derived from the interaction of atoms with a quantized field in a proper limit. The results of section 2 seem to suggest that this might only be possible if the system (atoms + field) is described as an open system, but the problem remains unresolved.

In section 3 we undertake a (partial) diagonalization of the semiclassical version of (1.1), with a view to studying the level statistics of the model. It is seen that, in spite of the anisotropy ( $\varepsilon \neq 1$ ) in (1.1), both the Hamiltonian (1.1) and its semiclassical version are partially diagonalizable by the same transformation introduced by Shore and Sander [10]. The eigenvalue equation is formulated in a subspace corresponding to a definite eigenvalue of the parity operator introduced by Graham and Höhnerbach (see [1] and references therein). In addition, due to the nature of the phase approximation and the semiclassical expansion, we must restrict ourselves to a finite subspace of  $\mathbb{C}^2 \otimes \mathcal{F}$ ; this leads us naturally to the study of boundary conditions. Both free and cyclic boundary conditions are considered, and it is seen that the latter must be chosen if unitarity of the phase operator is required, relating the present formulation to the nice construction by Pegg and Barnett [11]. In an appendix we briefly point out the parts of the relationship between the formalisms of [8] and [11] which are actually used in section 3. A more general discussion will be given elsewhere [12]. For completeness, we also briefly discuss in section 3 the effect of the Shore-Sander transformation on these boundary conditions.

In section 4 we present the nearest-neighbour distribution (NND) of levels, using the results of section 3 and varying the parameter  $\varepsilon$  from 0 to 1. The NND is seen to change from a 'delta-peak' distribution (for  $\varepsilon = 0$ ) to a distribution (for  $\varepsilon = 1$ ) similar to the one found by Kus [13], which is not of the general type associated with chaotic systems for reasons which are well understood [13]. The results are compared with the analogous ones for the original model (no phase approximation). The basic and most interesting result is that the two level statistics are generally qualitatively different (except for  $\varepsilon = 1$ ). This is further explained in section 4. The distribution of spacings between neighbouring levels depends, however, on level density correlations of all orders and is, therefore, a non-perturbative quantum effect. It is therefore of interest to know whether qualitative differences between the original and semiclassical models persist in the level statistics of finite order, in particular in the statistics of lowest order (bilinear) in the level density such as  $\Delta_3$ . This is not the case, however, as we show and discuss in section 4. We also show that for a higher value of the spin  $S$  ( $S = \frac{9}{2}$ )

there is a transition from the Poisson NND (for  $\varepsilon \approx 0$ ) to the Wigner GOE (for  $\varepsilon = 1$ ) in the original model (this was already known for  $\varepsilon = 1$  [14]). This result is a good demonstration of the effectiveness of the parameter  $\varepsilon$ , but we do not include it in section 4 because for such a value of  $S$  one seems to be close to the classical (as opposed to semiclassical) limit (see also [2] and [15]).

**2. The equations of motion in the phase representation of Bialynicki-Birula**

In this section we restrict ourselves, for simplicity, to the spin boson model ( $\varepsilon = 1$  in (1.1)), whereby the Hamiltonian becomes

$$H = \omega a^+ a + \omega_0 S_z + \lambda S_x (a + a^+). \tag{2.1}$$

Following [8], we introduce a representation of the eigenstates  $|n\rangle$  of the number operator  $N$  in terms of harmonic wavefunctions of an auxiliary variable  $\phi$ . It is convenient to ‘shift’ the eigenvalues by a constant integer  $n_0$  (interpreted as the ‘large’ photon number around which the state of the radiation field is expanded), adopting the following correspondence rule between state vectors and wavefunctions [8]:

$$|n_0 + m\rangle \rightarrow e^{im\phi}.$$

The corresponding representation of the annihilation and creation operators are given by

$$a = e^{-i\phi} (n_0 + p_\phi)^{1/2} \tag{2.2}$$

$$a^+ = (n_0 + p_\phi)^{1/2} e^{i\phi} \tag{2.3}$$

$$a^+ a = n_0 + p_\phi \tag{2.4}$$

where  $p_\phi \equiv -id/d\phi$ , acting on the Hilbert space  $L^2(0, 2\pi)$ . The above representation is clearly valid only on the subspace with  $n_0 + p_\phi \geq 0$  in order that the square roots in (2.3) and (2.4) be defined. This and other mathematical questions, as well as the relationship with the unitary phase introduced by Pegg and Barnett [11], will be treated elsewhere [12], but we provide a brief discussion in the appendix, where, in particular, the cyclic boundary condition used in section 3 is justified. For large  $n_0$ ,

$$a \approx \sqrt{n_0} e^{-i\phi} \left( 1 + \frac{1}{2n_0} p_\phi + \dots \right) = \sqrt{n_0} e^{-i\phi} + \frac{1}{2\sqrt{n_0}} e^{-i\phi} p_\phi + \dots \tag{2.5}$$

$$a^+ \approx \sqrt{n_0} \left( 1 + \frac{1}{2n_0} p_\phi + \dots \right) e^{i\phi} = \sqrt{n_0} e^{i\phi} + \frac{1}{2\sqrt{n_0}} p_\phi e^{i\phi} + \dots \tag{2.6}$$

The above expansions hold strongly on a proper subspace of  $\mathcal{F}$  (see appendix). It is often asserted that, in the phase representation, ‘in the limit  $n_0 \rightarrow \infty$ ’, the Hamiltonian may be written as

$$\tilde{H} = n_0 \omega + \omega p_\phi + \omega_0 S_z + 2\lambda \sqrt{n_0} S_x \cos \phi \tag{2.7}$$

([16]; see also [1] section 3c). The Heisenberg equations of motion for this Hamiltonian are

$$\dot{S}_x = i[\tilde{H}, S_x] = -\omega_0 S_y \tag{2.8a}$$

$$\dot{S}_y = i[\tilde{H}, S_y] = \omega_0 S_x - 2\lambda \sqrt{n_0} S_z \cos \phi \tag{2.8b}$$

$$\dot{S}_z = i[\tilde{H}, S_z] = 2\lambda \sqrt{n_0} S_y \cos \phi \tag{2.8c}$$

$$\dot{\phi} = i[\tilde{H}, \phi] = \omega \tag{2.8d}$$

$$\dot{p}_\phi = i[\tilde{H}, p_\phi] = 2\lambda \sqrt{n_0} S_x \sin \phi. \tag{2.8e}$$

Note that, by (2.8e), the equation for  $p_\phi$  does contain the 'back reaction' effect of the atoms. This effect does not, however, appear in the first-order approximation for the annihilation operator in (2.5):

$$a_1 = \sqrt{n_0} e^{-i\phi}$$

(we denote by a subscript the 'order of approximation'). Indeed, if we use the Hamiltonian  $\tilde{H}$  given by (2.7), the Heisenberg equation of motion for  $a_1$  reads

$$\dot{a}_1 = i[\tilde{H}, a_1] = \sqrt{n_0} i\omega [p_\phi, e^{-i\phi}] = -i\omega a_1 \quad (2.8f)$$

and similarly

$$\dot{a}_1^+ = i\omega a_1^+ \quad (2.8g)$$

Equations (2.8f) and (2.8g) together with (2.8a)–(2.8c) are just the equations of Autler and Townes ([9], [1] section 3c) corresponding to the non-autonomous effective Hamiltonian

$$H_{AT} \equiv \omega_0 S_z + 2\lambda\sqrt{n_0} S_x \cos \omega t.$$

They correspond to the (physically realizable) situation of no back reaction of the atoms on the field, expressed by (2.8f) and (2.8g), which may be compared with the Heisenberg equation of motion in the original model:

$$\dot{a} = i[H, a] = -i\omega a - i\lambda S_x \quad (2.9)$$

(and a similar equation for  $a^+$ ).

Note that, in order to obtain all the terms on the RHS of (2.9), one has to expand both  $H$  and  $a$  to order  $1/\sqrt{n_0}$ . We see, however, that in order to obtain (2.8f) it was necessary to keep the term  $\omega p_\phi$  in Hamiltonian (2.7). Note that this term, although exactly related to  $a^+ a$  by (2.4), actually results from the 'mixed term' in  $a^+ a$  in expansions (2.5) and (2.6) if we ask for a consistent development in descending powers of  $\sqrt{n_0}$ . Hence, we must use the expansions up to second-order terms given by (2.5) and (2.6) in the third (coupling) term of Hamiltonian (2.1). This requires, however, adding to (2.7) an additional term. Indeed, if we try to keep (2.7) and  $a$  in the form

$$a_2 \equiv \sqrt{n_0} e^{-i\phi} + \frac{1}{2\sqrt{n_0}} e^{-i\phi} p_\phi \quad (2.10)$$

then some 'back reaction' is necessarily present, but not in the correct form, i.e. the Heisenberg equation (2.9) is not preserved up to second order, as we now show. Using  $\tilde{H}$  given by (2.7) as the Hamiltonian, we have, by (2.10),

$$\begin{aligned} \dot{a}_2 &= i[\tilde{H}, a_2] = -i\omega\sqrt{n_0} e^{-i\phi} - \frac{i\omega}{2\sqrt{n_0}} e^{-i\phi} p_\phi + \lambda S_x e^{-i\phi} \sin \phi \\ &= -i\omega a_2 + \lambda S_x e^{-i\phi} \sin \phi \end{aligned}$$

which does not agree with (2.9). Now, replacing  $\tilde{H}$  by the 'second-order' Hamiltonian

$$H_2 \equiv n_0\omega + \omega p_\phi + \omega_0 S_z + 2\lambda\sqrt{n_0} S_x \cos \phi + \frac{\lambda}{2\sqrt{n_0}} S_x (e^{-i\phi} p_\phi + p_\phi e^{i\phi}) \quad (2.11)$$

we find

$$\begin{aligned} \dot{a}_2 &= i[H_2, a_2] \\ &= -i\omega\sqrt{n_0} e^{-i\phi} - \frac{i\omega}{2\sqrt{n_0}} e^{-i\phi} p_\phi + \lambda S_x e^{-i\phi} \sin \phi \\ &\quad + \frac{i\lambda}{2\sqrt{n_0}} S_x [e^{-i\phi} p_\phi + p_\phi e^{i\phi}, \sqrt{n_0} e^{-i\phi}] + O(n_0^{-1}) \\ &= -i\omega a_2 + \lambda S_x e^{-i\phi} \sin \phi + \frac{i\lambda}{2} S_x (e^{-i\phi} [p_\phi, e^{-i\phi}] + [p_\phi, e^{-i\phi}] e^{i\phi}) + O(n_0^{-1}) \\ &= -i\omega a_2 + \lambda S_x e^{-i\phi} \sin \phi - \frac{i\lambda}{2} S_x (e^{-2i\phi} + 1) + O(n_0^{-1}) \\ &= -i\omega a_2 - i\lambda S_x + O(n_0^{-1}) \end{aligned}$$

in agreement with (2.9).  $H_2$  leads to the Heisenberg equations of motion

$$\dot{S}_x = i[H_2, S_x] = -\omega_0 S_y \tag{2.12a}$$

$$\begin{aligned} \dot{S}_y &= i[H_2, S_y] = \omega_0 S_x - \lambda S_z \left( 2\sqrt{n_0} \cos \phi + \frac{1}{2\sqrt{n_0}} (e^{-i\phi} p_\phi + p_\phi e^{i\phi}) \right) \\ &= \omega_0 S_x - \lambda S_z (a_2 + a_2^\dagger) \end{aligned} \tag{2.12b}$$

$$\begin{aligned} \dot{S}_z &= i[H_2, S_z] = \lambda S_y \left( 2\sqrt{n_0} \cos \phi + \frac{1}{2\sqrt{n_0}} (e^{-i\phi} p_\phi + p_\phi e^{i\phi}) \right) \\ &= \lambda S_y (a_2 + a_2^\dagger) \end{aligned} \tag{2.12c}$$

which agree with the equations for (2.1). Further,

$$\dot{\phi} = i[H_2, \phi] = \omega + \frac{\lambda}{\sqrt{n_0}} S_x \cos \phi \tag{2.12d}$$

$$\dot{p}_\phi = i[H_2, p_\phi] = 2\lambda\sqrt{n_0} S_x \sin \phi + \frac{i\lambda}{2\sqrt{n_0}} S_x (e^{-i\phi} p_\phi - p_\phi e^{i\phi}). \tag{2.12e}$$

Hence, the additional operator term in (2.11) introduces a nonlinear coupling between the spin and boson degrees of freedom in (2.12a)–(2.12e), and back reaction is always present in a consistent semiclassical expansion of the electromagnetic field (compare with the introduction).

For the purpose of determining the level statistics, it is convenient to perform a partial diagonalization of the Hamiltonian. This is discussed in the next section.

### 3. Partial diagonalization of the Hamiltonian and boundary conditions

In order to search for the eigenvalues of Hamiltonian (1.1), it is convenient to perform a transformation introduced by Shore and Sander [10]. Here, we adapt this transformation to our context, formulating it in a different way. The essential tool is the unitary operator

$$U = S_z(\mathbb{1} + R) + iS_y(\mathbb{1} - R) \tag{3.1}$$

on the tensor product  $\mathbb{C}^2 \otimes \mathcal{F}$  where

$$R = (-1)^N \quad (3.2)$$

obeys

$$R^2 = \mathbb{1} \quad (3.3a)$$

$$\{R, a\} = \{R, a^+\} = 0. \quad (3.3b)$$

In (3.2),  $N$  is the number operator and in the above formula  $\{\cdot, \cdot\}$  denotes the usual anticommutator. Compared with [10], the transformation (3.1) is directly formulated here in terms of spin- $\frac{1}{2}$  operators, independently of the choice of a basis in  $\mathbb{C}^2$ . Performing the unitary transformation  $H' = UHU^{-1}$  on the Hamiltonian  $H$  given by (1.1) we obtain

$$H' = \omega a^+ a + \omega_0 S_z R + \frac{\lambda}{4} (1 + \varepsilon)(a + a^+) + \frac{\lambda}{2} (1 - \varepsilon) S_z R (a - a^+). \quad (3.4)$$

In the calculation leading to (3.4) we have only used well-known properties of the  $S_x$ ,  $S_y$ ,  $S_z$  operators and properties (3.3a)-(3.3b) of  $R$ . The advantage of (3.4) is that  $H'$  is diagonal in the spin coordinate. We also point out that the same transformation (3.1) works for all  $\varepsilon$  in (1.1).

Looking now for a semiclassical version of Hamiltonian (3.4), we employ the phase representation of Bialynicki-Birula and Bialynicki-Birula and, as in section 2, introduce in a consistent manner the second-order expansions (2.5) and (2.6) in (3.4). This procedure yields the following second-order semiclassical Hamiltonian corresponding to  $H'$ :

$$\begin{aligned} H'_2 = & \omega(n_0 + p_\phi) + \omega_0 S_z R + \frac{\lambda}{2} (1 + \varepsilon) \sqrt{n_0} \cos \phi + \frac{\lambda}{8\sqrt{n_0}} (1 + \varepsilon) (e^{-i\phi} p_\phi + p_\phi e^{i\phi}) \\ & + i\lambda(\varepsilon - 1) \sqrt{n_0} S_z R \sin \phi + \frac{\lambda}{4\sqrt{n_0}} (1 - \varepsilon) S_z R (e^{-i\phi} p_\phi - p_\phi e^{i\phi}) \end{aligned} \quad (3.5)$$

where  $R$  is given by (3.2) with  $N = n_0 + p_\phi$ . An expression identical to (3.5) is obtained by performing the Shore-Sander transformation (3.1) (with  $N = n_0 + p_\phi$  in (3.2)) directly on the semiclassical version (2.11) (more precisely, on the corresponding analogue of (2.11) when  $\varepsilon \neq 1$ ).

In order to compute the level statistics in section 4, we now formulate the eigenvalue problem in a subspace corresponding to a definite eigenvalue of the parity operator  $P = 2S_z R$  which is, up to a factor, the parity operator introduced by Graham and Höhnerbach (see [1] and references therein). It is easy to verify that  $P$  commutes individually with the four terms in the Hamiltonian (1.1). Therefore, we have  $[H, P] = 0$  for all  $\varepsilon$ . In particular, for  $\varepsilon \neq 0$ ,  $P$  is the only operator which commutes with  $H$ . Performing the transformation (3.1) on  $P$ , we obtain the very simple result

$$P' \equiv UPU^{-1} = 2S_z \quad (3.6)$$

and it follows that  $[H', P'] = 0$ .

The eigenvalue equation for the system with no phase approximation on a subspace of definite eigenvalue of  $P'$  may now be immediately written down from (3.4) and (3.6) using the basis  $|+\rangle \otimes |n\rangle$  (or  $|-\rangle \otimes |n\rangle$ ), where  $S_z |\pm\rangle = \pm \frac{1}{2} |\pm\rangle$ . The thus reduced Hamiltonians are of tridiagonal type and therefore easily diagonalizable numerically.

In the semiclassical approximation one proceeds similarly using (3.5) and (3.6) on the basis  $|+\rangle \otimes |n_0 + m\rangle$  (or  $|-\rangle \otimes |n_0 - m\rangle$ ). These equations, which we do not write down here explicitly, are the ones which will be used in section 4 to compute the level statistics in both cases.

The nature of the phase approximation implies that the base states used in the semiclassical case must satisfy  $n_0 + m \geq 0$  (because of the square roots in (2.2) and (2.3)). Further, the expansions (2.5) and (2.6) imply that one must require  $m < n_0$  (see the appendix). In practice,  $m$  will be taken such that  $-m_0 \leq m \leq m_0$  with  $n_0 \gg m_0$  in order that the second-order approximation used here will be sufficiently precise. Hence we must formulate the eigenvalue equation in a finite subspace; this is of course always necessary in a numerical treatment and we are thus led naturally to study the boundary conditions at  $m_0$  and  $-m_0$ .

We consider both free and cyclic boundary conditions defined by

$$\left. \begin{aligned} a^+ |n_0 + m_0\rangle &= 0 \\ a |n_0 - m_0\rangle &= 0 \end{aligned} \right\} \text{(free)} \quad (3.7)$$

$$\left. \begin{aligned} a^+ |n_0 + m_0\rangle &= \sqrt{n_0 - m_0} e^{-i(2m_0+1)\theta_0} |n_0 - m_0\rangle \\ a |n_0 - m_0\rangle &= \sqrt{n_0 - m_0} e^{i(2m_0+1)\theta_0} |n_0 + m_0\rangle \end{aligned} \right\} \text{(cyclic)} \quad (3.8)$$

with  $\theta_0 \in [0, 2\pi)$  being an arbitrary angle. Although the free boundary conditions are sometimes more convenient, as we shall see, the cyclic boundary conditions must be imposed to guarantee unitarity of the phase operator (see the appendix).

By (3.1) and (3.2) we see that in the case of free boundary conditions, the Shore-Sander transformed Hamiltonian (3.4) is preserved on the boundary states because properties (3.3a) and (3.3b) remain valid and are the only ones needed. This is, however, not the case for the cyclic boundary conditions because, instead of (3.3b), we have

$$\begin{aligned} \{R, a\} |n_0 - m_0\rangle &= 2Ra |n_0 - m_0\rangle \\ \{R, a^+\} |n_0 + m_0\rangle &= 2Ra^+ |n_0 + m_0\rangle. \end{aligned}$$

The above relations lead to extra boundary terms in the transformed Hamiltonian which do not even commute with  $P'$ .

Although the boundary conditions have no influence on global quantities such as the density of states and the level statistics of section 4, some local properties do depend on them. For example, the expectation values of  $[\widehat{\sin \phi}]^2$  and  $[\widehat{\cos \phi}]^2$  in an eigenstate of the number operator equal  $\frac{1}{2}$  with cyclic boundary conditions but not otherwise (see [11]).

#### 4. Level statistics with and without the phase approximation

In this section we compare the NND [17] for Hamiltonian (1.1) (original model, no phase approximation) with the NND for the same Hamiltonian with the phase approximation (2.10). By the results of section 3 this is equivalent to comparing the level statistics of Hamiltonians  $H'$ , given by (3.4), and  $H'_2$ , given by (3.5), which are both in tridiagonal form. The results are plotted in figures 1(a-f) (NND with no phase approximation) and figures 2(a-f) (NND with phase approximation). We have chosen



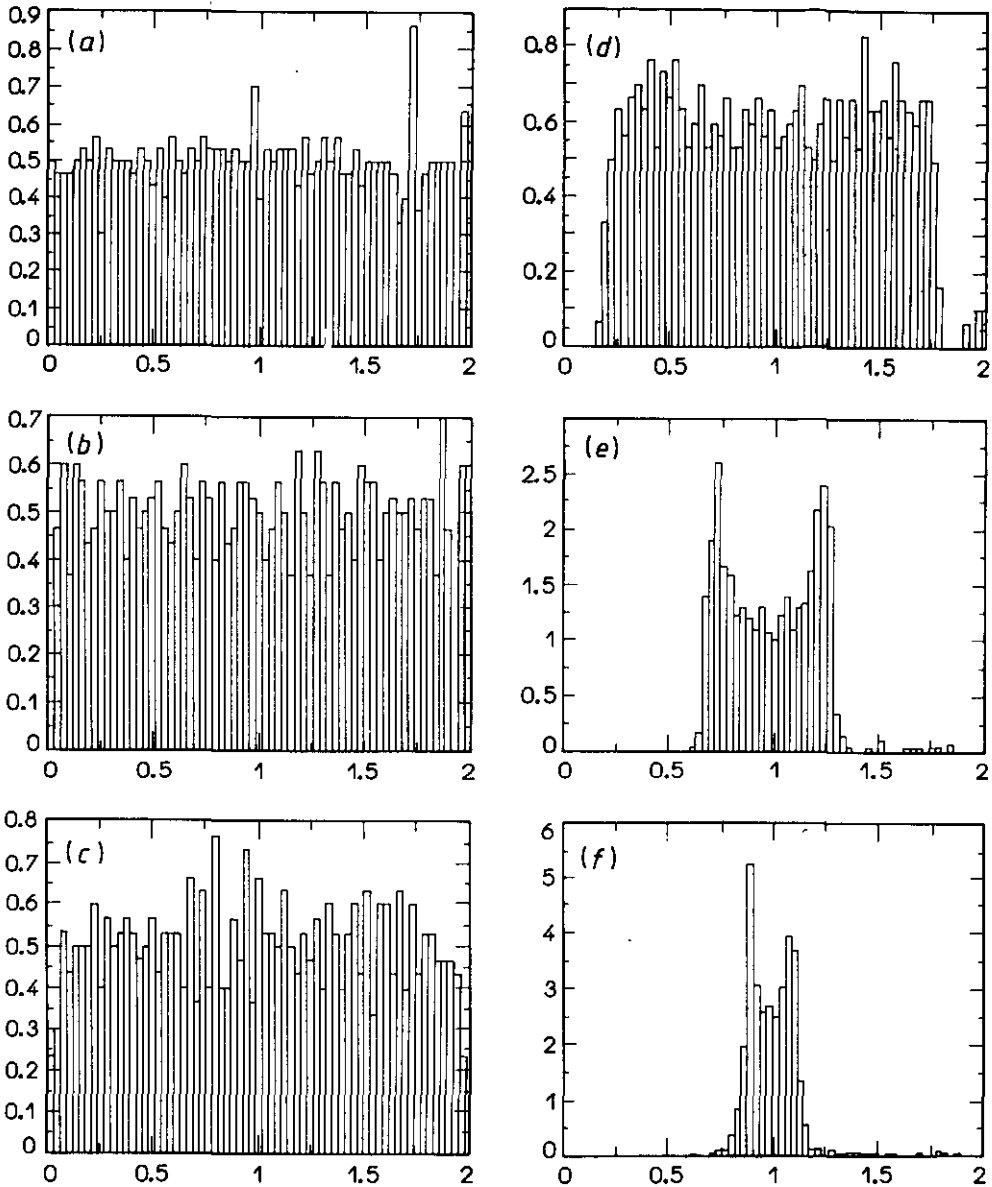


Figure 1. Nearest-neighbour spacing distribution of levels (original model, no phase approximation): (a)  $\varepsilon = 0$ , (b)  $\varepsilon = 0.3$ , (c)  $\varepsilon = 0.5$ , (d)  $\varepsilon = 0.7$ , (e)  $\varepsilon = 0.9$ , (f)  $\varepsilon = 1$ .

$\lambda = 1$ ,  $\omega = \omega_0 = 1$  and diagonalized a  $10^3 \times 10^3$  matrix in each case (in the case with phase approximation, we took  $n_0 = 10^6$  and the eigenvalues  $m$  of  $p_\phi$  ranging from  $-m_0$  to  $m_0$ , with  $m_0 = 500$ ). Each distribution in figures 1 and 2 corresponds to a value of  $\varepsilon$  (ranging from 0 to 1 and specified in the captions). Note that the level statistics have been calculated for the unfolded level distribution  $\rho(E)/\bar{\rho}(E)$ , where  $\rho(E)$  is the exact level density and  $\bar{\rho}(E)$  the average level density ([17], p 13).

What is conspicuous upon comparison is the difference between figures 1(a) and 2(a), both corresponding to the rotating-wave approximation ( $\varepsilon = 0$ ). In the original

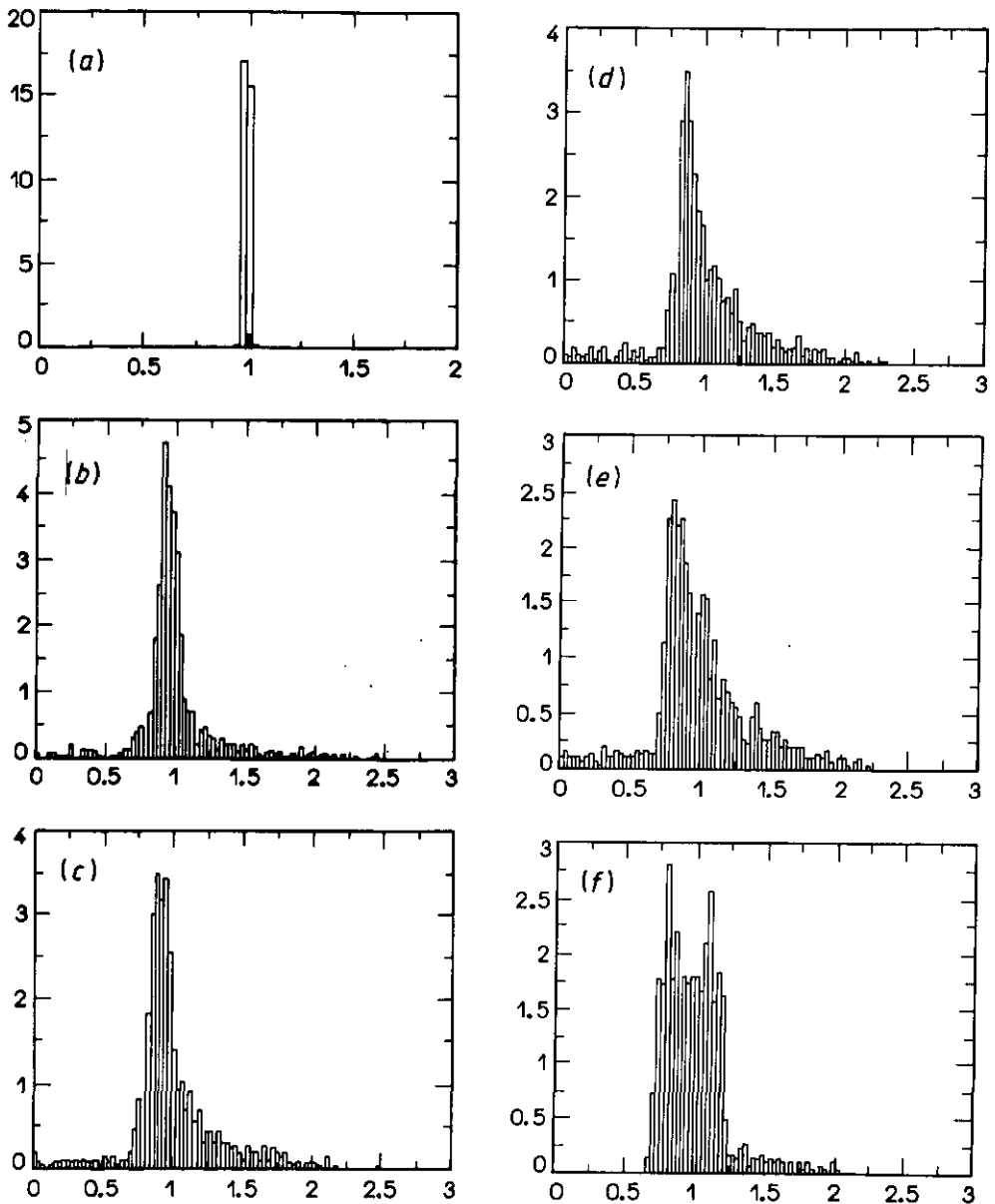


Figure 2. Nearest-neighbour spacing distribution of levels (semiclassical model, with phase approximation): (a)  $\epsilon = 0$ , (b)  $\epsilon = 0.3$ , (c)  $\epsilon = 0.5$ , (d)  $\epsilon = 0.7$ , (e)  $\epsilon = 0.9$ , (f)  $\epsilon = 1$ .

model (figure 1(a)) the NND is approximately uniform, while it is a 'delta peak' at  $\omega = \omega_0 = 1$  for the model with phase approximation. Varying  $\epsilon$ , the 'delta peak' becomes progressively blurred (figures 2(b-e)) but still different from the distribution for the original model (figures 1(b-e)), while for  $\epsilon = 1$  both distributions are very similar (figures 1(f) and 2(f)).

The explanation for the above, at first sight striking, behaviour is simple. For  $n_0$  large we may, for the purpose of computing the spectrum and the NND, disregard the second-order terms in (2.10) (we have verified this assertion by calculating the NND

with and without the terms proportional to  $1/\sqrt{n_0}$  in (3.5)). The resulting Hamiltonian obtained from (1.1) (with  $\varepsilon = 0$  and  $\omega = \omega_0 = 1$ ) is

$$H = n_0 + p_\phi + S_z + \lambda\sqrt{n_0}(S^+ e^{-i\phi} + S^- e^{i\phi}) \quad (4.1)$$

which has the additional conservation law

$$C = n_0 + p_\phi + S_z. \quad (4.2)$$

An eigenvector  $\psi$  of  $C$  corresponding to the eigenvalue  $n_0 + m + \frac{1}{2}$  of  $C$  is of the form

$$\psi = c_1|+\rangle \otimes |m\rangle + c_2|-\rangle \otimes |m+1\rangle \quad (4.3)$$

where  $p_\phi|m\rangle = m|m\rangle$ , and  $S_z|\pm\rangle = \pm\frac{1}{2}|\pm\rangle$ . The eigenvalue equation for  $\psi$  is of the simple form

$$(n_0 + m + \frac{1}{2})c_1 + \lambda\sqrt{n_0}c_2 = Ec_1$$

$$(n_0 + m + \frac{1}{2})c_2 + \lambda\sqrt{n_0}c_1 = Ec_2$$

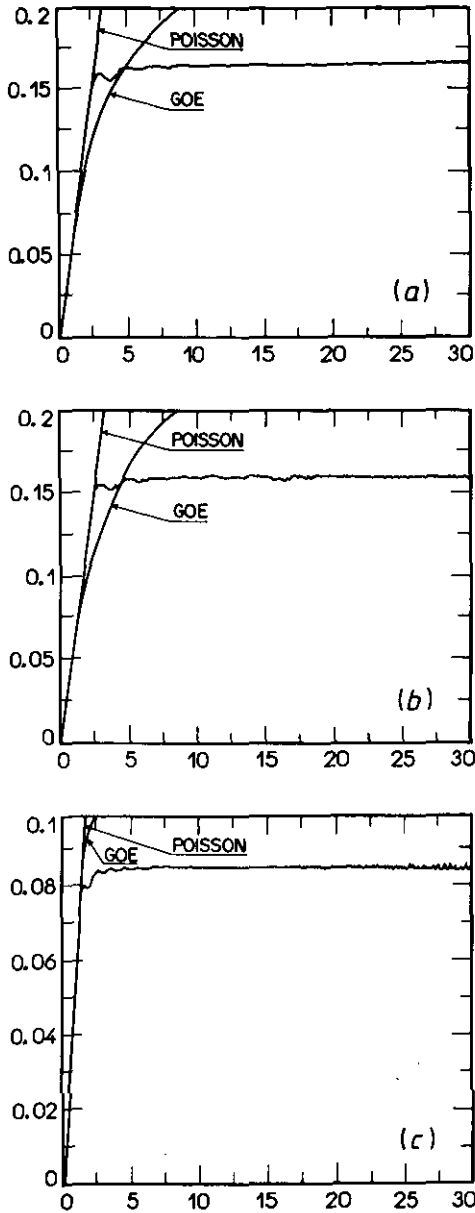
from which the eigenvalues follow:

$$E = n_0 + m + \frac{1}{2} \mp \lambda\sqrt{n_0}. \quad (4.4)$$

From (4.4) we see that, for  $n_0$  large, there is an overwhelming probability of having two neighbouring eigenvalues corresponding to two consecutive values of  $m$  in (4.4) and hence nearest-neighbour spacings equal to one, in agreement with figure 2(a). In contrast, for the original model the ' $\sqrt{n}$ ' and ' $\sqrt{n+1}$ ' factors coming from the operators  $a$  and  $a^+$  are 'uniformly distributed', leading to a uniform distribution in figures 1(a-e).

For  $\varepsilon = 1$  (and only in this case) the analysis of Kus ([14], see also [48]) shows that the NND is 'centred around' (see [18]) the NND for the integrable limit  $\lambda \rightarrow \infty$ . By (3.5) we see that  $\lambda \rightarrow \infty$  is—for the leading terms in (3.5) (i.e. disregarding the terms proportional to  $1/\sqrt{n_0}$  in (3.5))—equivalent to the limit  $n_0 \rightarrow \infty$ . This conclusion also holds, of course, if the phase approximation is performed directly upon the original Hamiltonian (1.1). This explains the similarity of figures 1(f) and 2(f).

The above discussion shows that, similarly to the classical limit [2], the semiclassical limit of the radiation field also entails special simplifications. The latter will probably turn out to be useful in several other applications to systems involving the electromagnetic field. Nevertheless, as remarked in the introduction, the NND is a non-perturbative quantum effect, and it is of interest to know whether a lower-order statistics also displays a qualitative difference between the original and semiclassical models. We have chosen the Dyson-Mehta  $\Delta_3$  statistics ([17], p 17). Figures 3 and 4 show that, as perhaps expected, no qualitative difference is visible at the level of  $\Delta_3$ . Indeed, figure 3 shows  $\Delta_3(L)$  as a function of  $L$  (see [17], p 17, for the definitions) in the cases  $\varepsilon = 0$ ,  $\varepsilon = 0.5$  and  $\varepsilon = 1$  for the original model, while figure 4 shows the same function with the same three values of  $\varepsilon$  for the semiclassical model (with phase approximation). We have set  $\hbar = 1$ . The limit  $\hbar \rightarrow 0$  is meaningless here because it would require the spin quantum number  $S \rightarrow \infty$ , but  $S$  is fixed equal to  $\frac{1}{2}$ . Although the Poisson and GOE curves are shown for comparison, they are not relevant to this model. In fact, in both the original and semiclassical models, after a value  $L \gg 1$ ,  $\Delta_3$  assumes a constant value typical of picket fence or harmonic oscillator statistics corresponding to a spectrum



**Figure 3.** Dyson-Mehta  $\Delta_3$  statistics (original model, no phase approximation): (a)  $\epsilon = 0$ , (b)  $\epsilon = 0.5$ , (c)  $\epsilon = 1$ .

consisting of a sequence of equally spaced levels ([17], p 17). The constant value  $\frac{1}{12} \approx 0.083$  agrees well with figure 3 for  $\epsilon = 1$  and with figure 4 for  $\epsilon = 0$ , and is not too far from the constant value in figure 4 for  $\epsilon = 0.5$  and  $\epsilon = 1$ , while it differs from the constant value in figure 3 for  $\epsilon = 0$  and  $\epsilon = 0.5$ . These facts may be explained as follows: in the semiclassical case the effective role of the quantum spin is reduced and the system resembles a harmonic oscillator throughout, while in the original model this is true for  $\epsilon = 1$  (for reasons discussed before) but for  $\epsilon \neq 1$  the role of the quantum spin

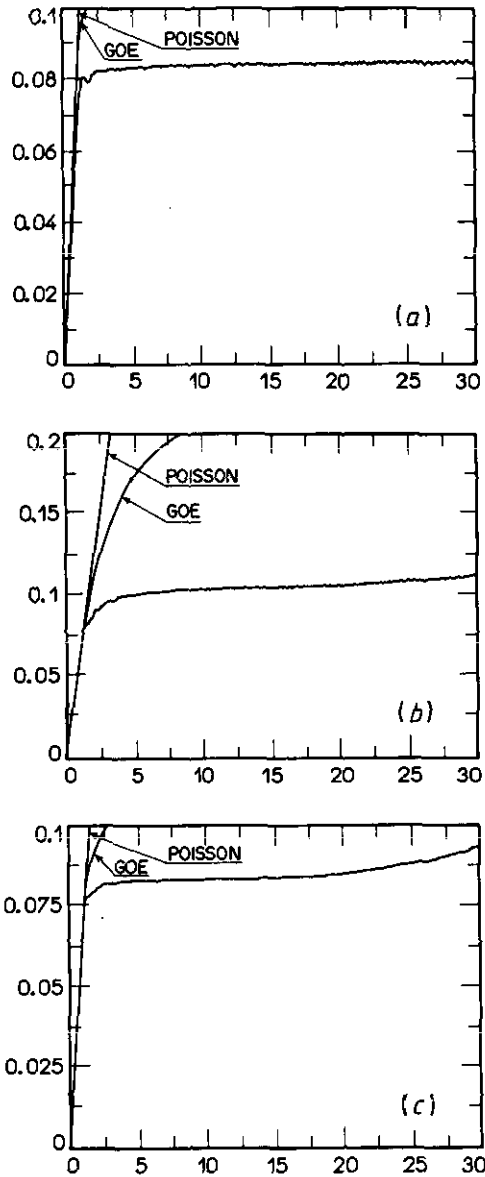


Figure 4. Dyson-Mehta  $\Delta_3$  statistics (semiclassical model, with phase approximation): (a)  $\varepsilon = 0$ , (b)  $\varepsilon = 0.5$ , (c)  $\varepsilon = 1$ .

is more strongly felt (although an additional conservation law is present in the rotating-wave approximation, the system is not 'equivalent' to a harmonic oscillator). A slight rise of  $\Delta_3$  in the semiclassical case (figure 4,  $\varepsilon = 0.5$  and  $\varepsilon = 1$ ) may be due to contributions of short classical orbits, but is not really understood. We have checked that for various higher values of the quantum spin  $S$ ,  $\Delta_3$  shows the behaviour corresponding to Poisson statistics for  $\varepsilon$  close to zero and to GOE for  $\varepsilon$  close to one, up to a saturation value  $L_{\max} = 2S + 1$ . For  $S = 1$  the behaviour is still of the harmonic oscillator type, and gradually changes. This will be discussed elsewhere [19].

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**Appendix**

In this appendix we provide a brief discussion of the relation between the phase operator of section 2 and the unitary phase operator recently introduced by Pegg and Barnett [11]. We show, in particular, that unitarity of the phase operator implies the cyclic boundary condition used in section 3, and that the thus defined operator coincides with the straightforward adaptation of the phase operator of [11] to the present situation. As a consequence, expansions (2.5) and (2.6) hold in the strong sense (i.e. on any state vector  $\psi$ ) on a proper subspace of  $\mathcal{F}$ . A more complete discussion will be given elsewhere [12].

By (2.2) and (2.3),

$$\exp(-i\phi)(n_0 + p_\phi) \exp(i\phi) - (n_0 + p_\phi) = 1 \tag{A1}$$

is equivalent to

$$[a, a^+] = 1. \tag{A2}$$

Now, usually, (A1) follows from

$$[\phi, p_\phi] = i \tag{A3}$$

but it is well known that (A3) (and the associated uncertainty relation) do not hold, in a domain  $\mathcal{D}$  where  $p_\phi$  is self-adjoint; in fact,  $\mathcal{D}$  consists of sufficiently smooth functions  $\psi \in L^2(0, 2\pi)$  such that  $\psi(0) = \psi(2\pi)$ ; however,  $\phi\psi$  is never in  $\mathcal{D}$  if  $\psi \in \mathcal{D}$  (see [20] for a complete discussion). Nevertheless, we only need (A1) for (A2) to be valid, and  $\exp(\pm i\phi)\psi \in \mathcal{D}$  if  $\psi \in \mathcal{D}$ , so that this difficulty does not arise here. However, definitions (2.2) and (2.3) imply that all vectors  $|m\rangle$  in the subspace of the definition satisfy  $n_0 + m \geq 0$  in order that the square roots in (2.2) and (2.3) are defined, and hence  $n_0 - s \geq 0$  if  $m \in [-s, s]$ . Now,  $\exp(-i\phi)$  is unitary only on the whole of  $L^2(0, 2\pi)$ . On the subspace generated by  $\{|m\rangle, m = -s, \dots, s\}$  one has to impose a cyclicity condition to achieve unitarity:

$$\exp[-i(s+1)\phi] = \exp(i\alpha) \exp(is\phi) \quad \alpha \in \mathbb{R}. \tag{A4}$$

Choosing  $\alpha = -(2s+1)\theta_0$  we get from (A4)

$$\phi_m = \theta_0 + \frac{2\pi m}{2s+1} \quad m = -s, \dots, s \tag{A5}$$

i.e. a discrete spectrum as in [11]. The thus defined unitary phase ‘ $\exp(-i\phi)$ ’ thus becomes equivalent to the following straightforward adaptation of the phase operator

$\widehat{\text{exp}}(i\phi)$  of [11] to the present situation. Define  $\mathcal{F}_{2s+1}$  as the subspace of  $\mathcal{F}$  generated by the states  $\{|n_0 - s\rangle, \dots, |n_0 + s\rangle\}$ ,  $s \leq n_0$ . Define 'phase states' by

$$|\gamma\rangle = (2s+1)^{-1/2} \sum_{n=n_0-s}^{n_0+s} \exp(in\gamma)|n\rangle \quad (\text{A6})$$

and let  $\phi_m$  be given by (A5). Then  $\{|\phi_m\rangle, m = -s, \dots, s\}$  form an orthonormal basis of  $\mathcal{F}_{2s+1}$  and a unitary phase operator  $\widehat{\text{exp}}(i\phi)$  is defined by

$$\widehat{\text{exp}}(i\phi)|\phi_m\rangle = \exp(i\phi_m)|\phi_m\rangle. \quad (\text{A7})$$

It follows [11] that

$$\begin{aligned} \widehat{\text{exp}}(i\phi) = & |n_0 - s\rangle\langle n_0 - s + 1| + |n_0 - s + 1\rangle\langle n_0 - s + 2| + \dots \\ & + |n_0 + s - 1\rangle\langle n_0 + s| + \exp[i(2s+1)\theta_0]|n_0 + s\rangle\langle n_0 - s|. \end{aligned} \quad (\text{A8})$$

Notice that  $\exp(-i\phi)$  acting on  $|m\rangle$  yields  $|m-1\rangle$ , in agreement with the action of  $\widehat{\text{exp}}(i\phi)$  on the basis of eigenfunctions of  $N$  which follows from (A8). Hence (2.2) corresponds to

$$a = \widehat{\text{exp}}(i\phi)N^{1/2}$$

and correspondingly

$$a^\dagger = N^{1/2} \widehat{\text{exp}}(-i\phi)$$

where  $\widehat{\text{exp}}(-i\phi) = [\widehat{\text{exp}}(i\phi)]^\dagger = [\widehat{\text{exp}}(i\phi)]^{-1}$ . The above operators do not, however, satisfy the relation  $[a, a^\dagger] = \mathbb{1}$ . Indeed, the relation corresponding to (A1) is

$$\widehat{\text{exp}}(i\phi)N \widehat{\text{exp}}(-i\phi) - N = \mathbb{1} - (2s+1)|n_0 + s\rangle\langle n_0 + s|. \quad (\text{A9})$$

However, on a subspace  $\mathcal{F}_{2s_0+1}$  with  $s_0 < s$  of 'physically accessible states' [11] (now from  $|n_0\rangle$  instead of  $|0\rangle$  in [11]), (A1) (and hence (A2)) holds; indeed,  $|n_0 + s\rangle\langle n_0 + s|$  in (A9) projects onto a state orthogonal to  $\mathcal{F}_{2s_0+1}$  if  $s_0 < s$ . The expansions (2.5) and (2.6) thus hold in the strong sense in  $\mathcal{F}_{2s_0+1}$ , if  $s_0 < n_0$ : choose, for example,  $s = n_0$  above; the commutation relation (A2) is then correct on the subspace  $\mathcal{F}_{2s_0+1}$  if  $s_0 < n_0$ . We now write (2.5) on  $\mathcal{F}_{2s_0+1}$ :

$$a = \sqrt{n_0} \exp(-i\phi) \left[ \mathbb{1} + \frac{1}{2n_0} p_\phi + \dots \right]. \quad (\text{A10})$$

Expansion (A10) holds strongly, i.e. on any  $\psi \in \mathcal{F}_{2s_0+1}$ , because, if  $\psi \in \mathcal{F}_{2s_0+1}$ ,  $\|p_\phi\psi\| \leq s_0\|\psi\|$ .

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